W-characteristic sets of LEX Gröbner bases

Preliminaries: LEX Gröbner bases

LEX term ordering \( u = x^n \) for \( \alpha \) if the left rightmost nonzero entry in the vector \( \alpha \) is positive.

\( \langle P \rangle \) is a Gröbner basis.

Normal Form \( \langle G \rangle \) is a Gröbner basis of \( (P) \).

Ideal \( \langle G \rangle \) is a Gröbner basis.

On W-characteristic Sets of LEX Gröbner Bases

Preliminaries: triangular sets

Variable ordering \( x_1 < \ldots < x_n \) for \( \langle P \rangle \) of \( F \)

Triangular set Any finite, nonempty ordered set \( T = \{T_{i_1}, \ldots, T_{i_n}\} \) of polynomials.

Pseudo-remainder \( F \) is a polynomial under \( \langle G \rangle \) two polynomials with \( \langle G \rangle = x_2 \).

Example (continued) \( x^2 + 2yz - (1 - a^2)z = \alpha \)

Preliminaries: triangular sets

Preliminaries: LEX Gröbner bases

Background: structures of LEX Gröbner bases

They were studied first by Lazard [4] for bivariate ideals and then extended to general zero-dimensional multivariate (radical) ideals. Based on the structures of LEX Gröbner bases, algorithms have been proposed to compute triangular decompositions out of LEX Gröbner bases for zero-dimensional ideals [5, 2]. The relationships between LEX Gröbner bases and Ritt characteristic sets were explored in [1] and then made clearer in [9] with the concept of W-characteristic sets.

W-characteristic set

Let \( P \subset \mathbb{K}[x] \) be a polynomial set and \( G \) be the reduced LEX Gröbner basis of \((P)\). Then the set

\[
\{ g \in \mathbb{K}[x] : g \not\in \langle G \rangle \} \cap \langle G \rangle < \mathbb{K}[x],
\]

ordered according to \( \langle \mathbb{K} \rangle \), where \( \langle \mathbb{K} \rangle = \{ g \in \mathbb{K}[x] : g(x) = x \} \), is called the W-characteristic set of \((P)\).

Basic properties

Let \( C \) be the W-characteristic set of \((P) \subseteq \mathbb{K}[x] \). Then (a) for any \( P \subseteq \mathbb{R} \), prem\((PC) \subseteq \mathbb{R} \) (b) \( C \cap \mathbb{K}[x] \subseteq \mathbb{P}[C] \subseteq \mathbb{Z}[C] \).

Normality and Pseudo-divisibility in W-characteristic sets (and thus in LEX Gröbner bases)

Either normality or pseudo-divisibility: a theorem

Let \( C = \{C_1, \ldots, C_l \} \) be the W-characteristic set of \((P) \subseteq \mathbb{K}[x] \). If \( C \) is not normal, then there exists an integer \( k \leq r \) such that \( \{C_1, \ldots, C_k \} \) is normal and \( \{C_{k+1}, \ldots, C_l \} \) is not regular.

Assume that the variables \( x_1, \ldots, x_r \) are ordered such that the parameters of \( C_i \) are all smaller than the other variables and let \( I_{k+1} \) be \( \text{in}(C_{k+1}) \), and \( I_k \) be the integer such that \( \text{lt}(I_k) = \text{lt}(C_k) \).

(a) If \( I_{k+1} \) is not reduced with respect to \( C_k \), then

\[
\text{prem}_{\langle C_{k+1} \rangle}(C_{k+1}) = I_k \text{ prem}_{\langle C_k \rangle}(C_k).
\]

(b) If \( I_{k+1} \) is reduced with respect to \( C_k \), then \( \text{res}(I_{k+1}, C_{k+1}) = 0 \) or \( \text{prem}_{\langle C_{k+1} \rangle}(C_{k+1}) = 0 \) as either \( \text{prem}_{\langle C \rangle}(C_{k+1}, C_{k+1}) = 0 \) or \( \text{prem}_{\langle C \rangle}(C_{k+1}, C_{k+1}) = 0 \).

Characteristic pairs

Either \( \text{prem}_{\langle C \rangle}(C_{k+1}, C_{k+1}) = 0 \) or \( \text{prem}_{\langle C \rangle}(C_{k+1}, C_{k+1}) = 0 \).

Example (continued)

The W-characteristic set above

\[
x^2 + 2yz - (1 - a^2)z - a = 0
\]

is not normal (the initial \( x^2 + z - a^2 \) involves \( x \)), and thus it is not regular:

\[
x^2 + 2yz - (1 - a^2)z = (x^2 + z - a^2)(x + 1),
\]

which corresponds to (b). left.

Characterization decomposition

A set \( \{G_1, \ldots, G_l\} \) of characteristic pairs of \((P) \subseteq \mathbb{K}[x] \) is called a characteristic decomposition of \((P) \) if

\[
Z(F) = \bigcup_{i=1}^l Z(G_i) \bigcup_{i=1}^l Z(\text{in}(G_i)) = \bigcup_{i=1}^l Z(\text{sat}(G_i)).
\]

If \( C \) is regular, \( C \) is normal and a Ritt characteristic set.

Transform to strong characteristic pair \( G \) is a characteristic pair, let \( \tilde{G} \) and \( \tilde{C} \) be the reduced LEX Gröbner basis and W-characteristic set of \( \text{sat}(C) \) respectively. Then \( \tilde{G} \) is normal, \( \tilde{C} \) is regular, and thus \( \tilde{G} \) is a strong characteristic pair.

References


